Q.2 a. If Z is a homogeneous function of degree n in x and y, show that

$$x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}=n(n-1)z$$

Answer: Page Number 221 of Text Book 1

b. Use method of differentiation under integral sign to show that

$$\int_0^\infty \frac{\tan^{-1}ax}{x(1+x^2)}dx=\frac{\pi}{2}\log(1+a),\ a\ge 0$$

Answer:

(b) Let $I(a)=\int_0^\infty \frac{\tan^{-1}ax}{x(1+x^2)}dx$

Differentiate under sign of integration w.r.t. a.

$$\frac{dI}{da}=\int_0^\infty \frac{1}{x(1+x^2)}\cdot\frac{x}{1+a^2x^2}dx \quad \text{---(1)}$$

$$=\frac{1}{(1-a^2)}\int_0^\infty\left(\frac{1}{1+x^2}-\frac{a^2}{1+a^2x^2}\right)dx \quad \text{---(2)}$$

$$=\frac{1}{1-a^2}\cdot\left[\tan^{-1}x-a\tan^{-1}ax\right]_0^\infty \quad \text{---(3)}$$

$$=\frac{1}{1-a^2}\left(\frac{\pi}{2}-a\frac{\pi}{2}\right)=\frac{\pi}{2}\frac{1}{1+a} \quad \text{---(4)}$$

Integrate w.r.t a.

$$I(a)=\frac{\pi}{2}\log(1+a)+c \quad ,\ a\ge 0$$

$I(0)=0=0+c \therefore c=0$. Hence $I(a)=\frac{\pi}{2}\log(1+a)$

Q.3 (a)

Q.3 a. Change the order of integration and then evaluate $\int_0^{4a}\int_{y^2/4a}^{2\sqrt{ay}} dx\,dy$

Answer: Page Number 298-299 of Text Book 1

b. Find the volume bounded by the xy-plane, the cylinder $x^2+y^2=1$ and the plane x+y+z = 3.

Answer:

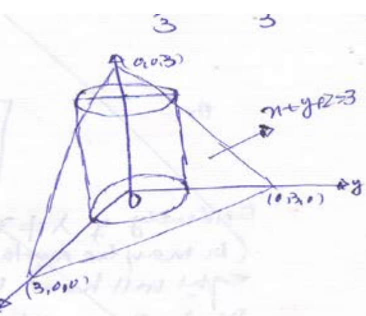
(b) Reqd. Volume $=\int_{-1}^{+1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}(3-x-y)\,dx\,dy$ —(2)

$$=\int_{-1}^{+1}\left[3y-xy-\frac{y^2}{2}\right]_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}dx$$

$$=\int_{-1}^{+1}\left[6\sqrt{1-x^2}-2x\sqrt{1-x^2}\right]dx$$

$$=\int_{-\frac{\pi}{2}}^{\pi/2}\left[6\cos\theta-2\sin\theta\cos\theta\right]\cos\theta\,d\theta \qquad \text{putting } x=\sin\theta \ \therefore dx=\cos\theta\,d\theta$$

$$=12\cdot\frac{1}{2}\frac{\pi}{2}+\left.2\frac{\cos^3\theta}{3}\right|_{-\pi/2}^{\pi/2}=3\pi$$



Q.4 a. Find the eigen values and eigen vectors of the matrix $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

Answer:

(a) Characteristic eqn of the given matrix is

$$|A-\lambda I| = \begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0 \quad \text{i.e. } \lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$$

eigen values of the given matrix are roots of characteristic eqn. These are 2, 2, 8

Eigen Vector $[x_1\ x_2\ x_3]$ corresponding to eigen value $\lambda = 2$ is given by

$$\begin{bmatrix} 6-2 & -2 & 2 \\ -2 & 3-2 & -1 \\ 2 & -1 & 3-2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \quad \text{ie} \begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

i.e. Sol. of $2x_1 - x_2 + x_3 = 0$ (remaining eqns are dependent on this)

Choosing $x_3 = 0$, $x_1 = 1$, $x_2 = 2$ ie eigenvector is $(1, 2, 0)$

Choosing $x_2 = 0$, $x_1 = 1$, $x_3 = -2$ ie eigen vector is $(1, 0, -2)$

These are two independent eigen vectors corresponding to two same eigen value $\lambda = 2$.

Similarly eigen vector corresponding to eigen value $\lambda = 8$ is given by

$$[A-8I]X = \begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \quad \text{or} \begin{bmatrix} -2 & -2 & 2 \\ 0 & -3 & -3 \\ 0 & -3 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$x_1 + x_2 - x_3 = 0$
i.e. $x_2 + x_3 = 0$
ie $x_3 = 1$
$x_2 = -1$ being
$x_1 = 2$

(b) The equations can be rewritten in matrix form

b. Determine the rank of the following matrices:

(i) $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$ (ii) $\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

Answer: Page Number 40 of Text Book 1

Q.5 a. Use Regula-Falsi method to compute real root of $xe^x = 2$ correct to three decimal places.

Answer:

(5) Rewrite the eqn as $f(x) = xe^x - 2 = 0$

$\therefore f(0) = -2$

$f(1) = e - 1 = .71828182\mathbf{8}$

f(0) and f(1) differ by sign. Hence root lies between 0 and 1.

$\therefore$ By Regula-falsi method, 1st approx. root is given by

$$x_1 = a - \frac{b-a}{f(b)-f(a)} \cdot f(a) = 0 - \frac{1-0}{f(1)-f(0)} \cdot f(0) = \frac{-1}{2.718281828}(-2)$$

$$= +0.73575888$$

$f(x_1) = f(.73575888) = .73575888 \cdot e^{.73575888} - 2 = -.46442328$

$\therefore$ Root lies between .73575888 and 1.0

By regula-falsi method second approx. root is given by

$$x_2 = \frac{a f(b) - b f(a)}{f(b) - f(a)} = \frac{.73575888 \times .718281828 + 1 \times .46442328}{.718281828 + .46442328}$$

$$= .83952077$$

$f(x_2) = -.0562935$

$\therefore$ Root lies between .83952077 and 1.0. By Regula-falsi method 3rd approximate root is

$$x_3 = \frac{x_2 f(x_1) - x_1 f(x_2)}{f(x_1) - f(x_2)} = \frac{.83952077 \times .71828 1828 + 1.0 \times 0.0562935}{.71821828 + 0.0562935}$$

$$= .85118385$$

$f(x_3) = -.00617137$

$\therefore$ Root lies between .85118385 and 1.0. By Regula-falsi method

$$x_4 = \frac{x_3 f(2) - x_2 f(x_3)}{f(2) - f(x_3)} = .852451567$$

$f(x_4) = -.00066888$

Hence correct answer upto 3 decimal places is .852

b. Find by Runge-Kutta method of order four, an approximate value of y at x = 0.2 for the equation $\frac{dy}{dx} = \frac{y-x}{y+x}, y(0) = 1$. Take h = 0.2.

Answer:

Here $\frac{dy}{dx} = f(x,y) = \frac{y-x}{y+x}$, $x_0 = 0$, $y_0 = 1$, $h = 0.2$

$\therefore K_1 = hf(x_0, y_0) = .2\,\frac{1-0}{1+0} = .2$

$K_2 = hf(x_0+\frac{h}{2}, y_0+\frac{K_1}{2}) = .2 \times \frac{(1+\frac{.2}{2})-(0+\frac{.2}{2})}{(1+\frac{.2}{2})+(0+\frac{.2}{2})} = .2 \times \frac{1}{1.2} = 0.166666$

$K_3 = hf(x_0+\frac{h}{2}, y_0+\frac{K_2}{2}) = .2 \times \frac{(1+\frac{.1666}{2})-(0+\frac{.2}{2})}{(1+\frac{.1666}{2})+(0+\frac{.2}{2})} = .2 \times \frac{.98333}{1.18333} = 0.166197$

$K_4 = hf(x_0+h, y_0+k_3) = .2 \times \frac{1.166197-.2}{1.166197+.2} = .2\,\frac{.966197}{1.366197} = 0.141443$

$K = \frac{K_1+2K_2+2K_3+K_4}{6} = \frac{.2+.333332+.332394+.141443}{6} = \frac{1.007169}{6} = .1678615$

Hence $y(.2) = y_0+K = 1.1678615$

Q.6a. Solve the differential equation $(x+1)\frac{dy}{dx}+1 = 2e^{-y}$

Answer:

Eqn. can be rewritten as

$(x+1)\frac{dy}{dx} = 2e^{-y}-1$

Separating Variables $\frac{dy}{2e^{-y}-1} = \frac{dx}{x+1}$ or $\frac{e^{y}\,dy}{2-e^{y}} = \frac{dx}{x+1}$

Integrating, $-\log(2-e^{y}) - \log(x+1) = \text{const}$

or $(2-e^{y})(x+1) = \text{const}$ Ans.

b. Find the orthogonal trajectories of family of curves $ay^2 = x^3$

Answer:

Differentiating $ay^2 = x^3$

$a\,2y\,\frac{dy}{dx} = 3x^2$

$\therefore$ ~~Diff. eqn. of curves~~ Eliminate parameter 'a'

$\frac{x^3}{y^2}\cdot 2y\,\frac{dy}{dx} = 3x^2$ — ①

$\therefore$ Diff. Eqn. of orthogonal trajectory is (replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$)

$\frac{2x^3}{y}\left(-\frac{dx}{dy}\right) = 3x^2$ — ②

or $2x\,dx + 3y\,dy = 0$

Integrating $2x^2+3y^2 = \text{const}$ is reqd orth. traj.

Q.7 a. Solve the differential equation $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 25y = e^{2x} + x + \text{Sin}x$

Q.7 (a) A.E. of the given Diff. Eqn. is

$m^2 - 6m + 25 = 0 \quad \therefore m = 3 \pm 4i$

$\therefore$ C.F. $= e^{3x}(c_1 \cos 4x + c_2 \sin 4x)$

Particular Integral is given by

$$P.I. = \frac{1}{D^2 - 6D + 25}\left(e^{2x} + x + \sin x\right)$$

$$= \frac{1}{D^2 - 6D + 25} e^{2x} + \frac{1}{25}\left(1 - \frac{6D}{25} + \frac{D^2}{25}\right)^{-1} x + \frac{1}{D^2 - 6D + 25}\sin x$$

$$= \frac{1}{17}e^{2x} + \frac{1}{25}\left(1 + \frac{6D}{25} + \cdots\right)x + \frac{1}{-1 - 6D + 25}\sin x$$

$$= \frac{1}{17}e^{2x} + \frac{1}{25}\left(x + \frac{6}{25}.1\right) - \frac{1}{6}\frac{1}{D-4}\sin x$$

$$= \frac{1}{17}e^{2x} + \frac{1}{25}\left(x + \frac{6}{25}\right) - \frac{1}{6}\frac{1}{-1-16}(D+4)\sin x$$

$$= \frac{e^{2x}}{17} + \frac{1}{25}\left(x + \frac{6}{25}\right) + \frac{1}{102}(\cos x + 4\sin x)$$

Hence Complete sol is

$$y = e^{3x}(c_1 \cos 4x + c_2 \sin 4x) + \frac{1}{17}e^{2x} + \frac{1}{25}\left(x + \frac{6}{25}\right) + \frac{1}{102}(\cos x + 4\sin x)$$

b. Solve the equation $\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} = 12\frac{\log x}{x^2}$

Answer:

(b) Given eqn can be rewritten as

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}=12\log x \quad \text{—(1)}$$

Put $x=e^t$ i.e. $t=\log x$, so that $x\frac{dy}{dx}=Dy$, $x^2\frac{d^2y}{dx^2}=D(D-1)y$

$\therefore$ (1) becomes where $D=\frac{d}{dt}$

$$D(D-1)y+Dy=12t$$

or $$D^2y=12t \quad \text{—(2)}$$

Its complementary function $=c_1+c_2t$

and $$P.I=\frac{1}{D^2}12t=12\frac{t^3}{2\times 3}=2t^3$$

Hence complete soln

$$y=c_1+c_2t+2t^3=c_1+c_2\log x+2(\log x)^3$$

Q.8 a. Obtain the series solution of $\left(1-x^2\right)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+2y=0$

Answer:

Q.8 (a) Here $x=0$ is an ordinary point because coefft of $y''\neq 0$ at $x=0$

$\therefore$ Let

$$y=a_0+a_1x+a_2x^2+a_3x^3+\cdots \quad \text{—(1)}$$

$$\therefore \quad y'=a_1+2a_2x+3a_3x^2+4a_4x^3+\cdots$$

$$y''=2a_2+3.2a_3x+4.3a_4x^2+5.4a_5x^3+\cdots$$

Putting these values in the given equation,

$$(1-x^2)[2a_2+3.2a_3x+4.3a_4x^2+5.4a_5x^3+\cdots]$$
$$-2x(a_1+2a_2x+3a_3x^2+4a_4x^3+\cdots)+2(a_0+a_1x+a_2x^2+a_3x^3+\cdots)=0$$

equating coeffts of different powers of x on both sides, we get

$2a_2+2a_0=0$ i.e. $a_2=-a_0$ (const terms)

$3.2a_3-2a_1+2a_1=0$ i.e. $a_3=0$ (coeffts of x)

$4.3a_4-\cancel{2}a_2-4a_2+\cancel{2}a_2=0$ i.e. $a_4=\frac{a_2}{3}=-\frac{a_0}{3}$ (coeffts of x^2)

$5.4a_5-3.2a_3-2.3a_3+2a_3=0$ i.e. $a_5=\frac{10}{20}a_3=0$

etc

Putting values in (1), we get

$$y=a_0+a_1x-a_0x^2+0.x^3-\frac{a_0}{3}x^4+0x^5+\cdots$$
$$=a_0\left(1-x^2-\frac{1}{3}x^4-\cdots\right)+a_1x \quad \text{as reqd. sol.}$$

b. Show that $\beta(m,n)=\dfrac{\overline{|m}\,\overline{|n}}{\overline{|m+n}}$

Answer: Page Number 854 of Text Book 1

Q.9 a. State and prove Rodrigue's formula.

Answer:

9 (b) Legendre's polynomial $P_n(x)$ is given by

$$P_n(x) = \frac{1}{2^n \underline{|n}}\, \frac{d^n}{dx^n}(x^2-1)^n \quad —(1)$$

To prove (1), let

$$y=(x^2-1)^n$$

$$\therefore \frac{dy}{dx} = y_1 = n(x^2-1)^{n-1}\cdot 2x = \frac{2nx(x^2-1)^n}{(x^2-1)} = \frac{2nyx}{(x^2-1)}$$

$$\therefore (x^2-1)y_1 - 2nyx = 0 \quad —(2)$$

Differentiate (2) (n+1) times using Leibnitz's theorem

$$(x^2-1)y_{n+2} + (n+1)y_{n+1}\cdot 2x + \frac{(n+1)n}{\underline{|2}}y_n\cdot 2 - 2n[xy_{n+1} + (n+1)y_n\cdot 1] = 0$$

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$$(x^2-1)y_{n+2} + (2n+2-2n)xy_{n+1} - n(n+1)y_n = 0$$

or $(1-x^2)y_{n+2} - 2xy_{n+1} + n(n+1)y_n = 0 \quad —(3)$

If we put $V = y_n = \dfrac{d^n y}{dx^n} = \dfrac{d^n}{dx^n}(x^2-1)^n$, (3) becomes

$$(1-x^2)\frac{d^2V}{dx^2} - 2x\frac{dV}{dx} + n(n+1)V = 0$$

Which is Legendre's equation having $V = y_n = \dfrac{d^n y}{dx^n}$ as its solution

Or its constant multiple as solution. But its finite series solution is $P_n(x)$, a Legendre's polynomial.

$$\therefore P_n(x) = cV = cy_n = c\frac{d^n}{dx^n}(x^2-1)^n \quad —(4)$$

$$= c\frac{d^n}{dx^n}[(x-1)^n(x+1)^n]$$

To evaluate c, put $x=1$ and remember $P_n(1)=1$,

$1 = c[\underline{|n}\,(x+1)^n +$ terms containing various powers of $(x-1)$ which become zero at $x=1]$

$$\therefore c = \frac{1}{2^n\underline{|n}}$$

Hence $P_n(x) = \dfrac{1}{2^n\underline{|n}}\dfrac{d^n}{dx^n}(x^2-1)^n$

b. Prove that $J_{n+1}(n)+J_{n-1}(n)=\frac{2n}{x}J_n(n)$

Answer:

(b) We know that

$$\frac{d}{dx}\left(x^n J_n(x)\right) = x^n J_{n-1}(x) \quad —(1)$$

and $$\frac{d}{dx}\left(x^{-n} J_n(x)\right) = -x^{-n} J_{n+1} \quad —(2)$$

Denoting $\frac{d}{dx} J_n(x) = J_n'(x)$, (1) + (2) can be rewritten as

$$x^n J_n'(x) + n x^{n-1} J_n(x) = x^n J_{n-1}(x) \quad —(3)$$

$$x^{-n} J_n'(x) - n x^{-n-1} J_n(x) = -x^{-n} J_{n+1} \quad —(4)$$

or

$$J_n'(x) + \frac{n}{x} J_n(x) = J_{n-1}(x) \quad —(5)$$

$$J_n'(x) - \frac{n}{x} J_n(x) = -J_{n+1}(x) \quad —(6)$$

Substracting (6) from (5),

$$\frac{2n}{x} J_n(x) = J_{n-1}(x) + J_{n+1}(x)$$

Ans.

TEXT BOOKS

1. Higher Engineering Mathematics, Dr. B.S.Grewal, 40th edition 2007, Khanna publishers, Delhi.

2. Text book of Engineering Mathematics, N.P. Bali and Manish Goyal, 7th Edition 2007, Laxmi Publication (P) Ltd.